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On Damage-Spreading Transitions

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We study the damage-spreading transition in a generic one-dimensional stochastic cellular automaton with two inputs (Domany–Kinzel model). Using an original formalism for the description of the microscopic dynamics of the model, we are able to show analytically that the evolution of the damage between two systems driven by the same noise has the same structure as a directed percolation problem. By means of a mean-field approximation, we map the density phase transition into the damage phase transition, obtaining a reliable phase diagram. We extend this analysis to all symmetric cellular automata with two inputs, including the Ising model with heat-bath dynamics.

KEY WORDS: Damage spreading; directed percolation; stochastic cellular automata; disordered systems; symmetry breaking.

1. INTRODUCTION

In this paper we deal with the problem of the evolution of two replicas of a Boolean system (cellular automaton) that evolve stochastically under the same realization of the noise. The system is defined on a regular lattice of L sites and evolves in discrete time steps. We limit the explicit analysis to one-dimensional systems, but the results can be extended to higher dimensions.

Let us indicate the time with the index $t = 1, \ldots, \infty$ and the space with $i = 0, 1, \ldots, L-1$. All the operations on the space index i are assumed to be modulo L. The state variables $\sigma(i, t)$ can assume the values 0 or 1 (Boolean variables). The evolution of $\sigma(i, t)$ is given by probabilistic transition rules and depends on a small number of neigboring sites; in its simplest form, $\sigma(i, t)$ depends only on the state of the two nearest neigbors.

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In order to simplify the notation, we write $\sigma_+ = \sigma(i+1, t)$, $\sigma_- = \sigma(i-1, t)$, $\sigma' = \sigma(i, t+1)$. The evolution rule can be written as

$$\sigma' = f(\sigma_-, \sigma_+)$$

Since the number of possible values of the couple (σ_-, σ_+) is four, the function f is usually specified by giving the four transition probabilities $\tau(\sigma_-, \sigma_+ \to 1)$ from each possible configuration to one:

$$\begin{aligned} \tau(0, 0 \to 1) &= p_0 \\ \tau(0, 1 \to 1) &= p_1 \\ \tau(1, 0 \to 1) &= p_2 \\ \tau(1, 1 \to 1) &= p_3 \end{aligned} \tag{1}$$

The normalization condition gives $\tau(\sigma_-, \sigma_+ \to 0) = 1 - \tau(\sigma_-, \sigma_+ \to 1)$.

All the sites of the lattice are generally updated synchronously. Except for deterministic cellular automata, for which the transition probabilities are either zero or one, we do not expect strong differences between parallel and sequential updating.

This schematization naturally arise in the modeling of several systems (contact processes) in physical and biological investigations. It was introduced by Domany and Kinzel[1,2] and can be considered the prototype for all local stochastic processes. For a short review of the applicability of this model, see refs. 3 and 4.

In the thermodynamic limit, the Domany–Kinzel (DK) model exhibits a phase transition from an ordered to a disordered phase for $p_0 = 0$. The ordered configuration is $\sigma(i) = 0$ for all i (adsorbing state). The order parameter is the asymptotic density $m = \lim_{t \to \infty} \lim_{L \to \infty} m(t, L)$, where

$$m(t, L) = \frac{1}{L} \sum_{i=0}^{L-1} \sigma(i, t)$$

In the following, the symbol α will refer to the critical surface that separates the region $m = 0$ from $m > 0$.

This transition has been studied mainly for the symmetric case $p_1 = p_2$. Except for a phenomenological renormalization study,[5] the transition line has been found numerically to belong to the universality class of directed percolation, which is a particular case of the model. The disagreement for the renormalization group results can originate from finite-size effects. For the asymmetric case $p_1 \neq p_2$, it has been claimed[6] that the phase transition belongs to a different universality class (mean field).

The existence of an adsorbing state is a nonequilibrium feature of the model, allowing the presence of a phase transition also in a one-dimensional (spatial) system. It is shown is Section 4 that in the DK model there can be two adsorbing states, $\sigma(i)=0$ and $\sigma(i)=1$, related by a simple transformation of the transition probabilities. The two transition lines meet at the point $M(p_1=1/2, p_3=1)$. This point corresponds to the problem of a random walk in one dimension, and thus exhibits mean-field exponents.

A powerful tool for the investigation of this model is the study of damage spreading. One considers two replicas σ and η of the same model with different initial conditions (they can be completely uncorrelated or differ only in some sites). The two replicas evolve under the same realization of the stochasticity. The difference at site i and at time t between the two configurations is given by

$$h(i, t)=\sigma(i, t)\oplus\eta(i, t)$$

where the symbol $\oplus$ represents the sum modulo two (exclusive OR, XOR). Since we use Boolean variables ($a, b\in\{0, 1\}$), one can interpret the exclusive-or as $a\oplus b=a+b-2ab$. When mixing XOR and AND (represented as a multiplication), one can use the algebraic rules for the sum and the multiplication.

The order parameter for the damage-spreading transition is the asymptotic Hamming distance $H=\lim_{t\to\infty}\lim_{L\to\infty}H(t, L)$ defined as

$$H(t, L)=\frac{1}{L}\sum_{i=0}^{L-1}h(i, t)$$

using the usual sum.

The critical surface that separates the region $H=0$ from $H>0$ is indicated with the symbol γ.

In the DK model, numerical and analytical investigations[6–10, 3] indicated the existence of a damage spreading phase.

The damage phase transition can be thought of as an ergodicity-breaking transition: in the phase where the damage disappears, all initial conditions asymptotically follow a trajectory that does not depend on the initial conditions, but only on the realization of the noise.

The critical exponents for the density and the damage transitions in the plane $(p_1=p_2, p_0=0)$ are numerically the same.[6, 3] It has been conjectured[11, 12] that all continuous transitions from an adsorbing to an active state belong to the universality class of the DK model (and thus of directed percolation), and that the same universality class should include all damage-spreading transitions.[3]

Here we want to investigate the connection between the density phase transition and the damage phase transition in the DK model. We have to carefully describe the dynamics of the model: the position of the transition line depends on the way in which the randomness is implemented in the actual simulations. In Section 2 we introduce the formalism that allows an exact description of how randomness is implemented in the model. We are thus able to write down the evolution equation for the spins and to obtain the evolution equation for the distance between two replicas. The structure of the latter equation corresponds to the DK model with $p_0 = 0$. We conclude that the universality class of damage spreading is, at least for this simple case, that of directed percolation. In Section 3 we obtain the phase diagram of the DK model by mapping the transition line for the density to the transition line for the damage by means of mean-field approximations. In Section 4 we show that one can infer the existence of a phase transition for the damage also in cases for which there is no phase transition for the density and that there are two disjoint regions in the parameter space for the damage spreading. Finally, we give conclusions and open questions in the last section.

2. THE DAMAGE-SPREADING TRANSITION

Let us start from a simple example, the dilution of rule 90 (in Wolfram's notation)[13] that will also serve to fix the notation. Rule 90 is a deterministic rule that evolves according to

$$\sigma' = \sigma_- \oplus \sigma_+$$

The transition probabilities for the diluted rule 90 are

$$\tau(0, 0 \to 1) = 0$$

$$\tau(0, 1 \to 1) = p$$

$$\tau(1, 0 \to 1) = p$$

$$\tau(1, 1 \to 1) = 0$$

where p is the control parameter of the model.

In order to apply rule 90 for a fraction p of sites and rule 0 (all configurations give 0) for the rest, one usually extracts a random number $r = r(i, t)$ for each site and at each time step and chooses the application of rule 90 or rule 0 according to $r < p$ or $r \geqslant p$, respectively.

We can easily write the explicit expression for this rule by means of the function $[\,\cdot\,]$, assuming that $[\,\textit{logical proposition}\,]$ takes the value 1 if *logical*

proposition is true and 0 otherwise (this interpretation of logical propositions is the standard one in C language). Finally, we have for the diluted rule 90

$$\sigma' = [r < p](\sigma_- \oplus \sigma_+) \tag{2}$$

From a different point of view, one can extract all random numbers $r(i, t)$ before the simulation and attach them to the sites of the space-time lattice, regardless of their usage. The random numbers are thus similar to a space-time quenched (disordered) field.

Once given the set of random numbers, the evolution is completely deterministic, and the evolution function depends on the lattice position (spatial and temporal) via the random numbers $r(i, t)$. One can alternatively define the model stating that some deterministic functions are randomly distributed on the space-time lattice according to a certain probability distribution. This description is very reminiscent of the Kauffman model.[14]

The damage spreading can be considered a measure of the stability of the set of possible trajectories, averaging over the realizations of the noise. The (maximum) Lyapunov exponent is a measure of the instantaneous effects of a vanishing perturbation on a trajectory of a dynamical system. Since the state variables of cellular automata assume only integer values, one has to extend the definition to a finite initial distance (and to finite time steps), thus taking into account the possibility of nonlinear effects. For cellular automata, the smallest initial perturbation corresponds to a difference of only one site between the two replicas. The short-time effects of a (vanishing) perturbation define the analog of the derivatives for a continuous system.[15] The study of the equivalent of the usual (linear) Lyapunov exponent for deterministic cellular automata allows a classification of the rules according to the trend of the damage.[16] The general problem of damage spreading can thus be considered equivalent to the study of the nonlinear Lyapunov exponent (i.e., finite initial distance and finite evolution times) for space-time-disordered cellular automata.

Using the concept of Boolean derivatives,[15] we develop a Boolean function $f(a, b)$ as

$$f(a, b) = f_0 \oplus f_1 a \oplus f_2 b \oplus f_3 ab$$

where the Taylor coefficients are

$$f_0 = f(0, 0)$$

$$f_1 = f(0, 1) \oplus f(0, 0)$$

$$f_2 = f(1, 0) \oplus f(0, 0)$$

$$f_3 = f(1, 1) \oplus f(0, 1) \oplus f(1, 0) \oplus f(0, 0)$$

One can verify the previous expression by enumerating all the possible values of a and b.

Using the bracket $[\cdot]$ notation, we have that the transition probabilities (1) correspond to

$$f(0,0) = [r_0 < p_0]$$
$$f(0,1) = [r_1 < p_1]$$
$$f(1,0) = [r_2 < p_2]$$
$$f(1,1) = [r_3 < p_3]$$

where the random numbers $r_j(i,t)$ belong to the interval $[0,1)$ and constitute the quenched random field. We neglect to indicate the space and time indices for simplicity.

The Taylor coefficients become

$$f_0 = [r_0 < p_0]$$
$$f_1 = [r_1 < p_1] \oplus [r_0 < p_0]$$
$$f_2 = [r_2 < p_2] \oplus [r_0 < p_0]$$
$$f_3 = [r_3 < p_3] \oplus [r_2 < p_2] \oplus [r_1 < p_1] \oplus [r_0 < p_0]$$

In the following we shall assume $p_1 = p_2$ and $r_1 = r_2$, so that $f_1 = f_2$ and

$$f_3 = [r_3 < p_3] \oplus [r_0 < p_0]$$

The correlations among the random numbers r_j (at same space-time position) affect the position of the damage critical surface γ, as pointed out by Tomé,[10] Grassberger,[3] and Domany,[17] but not the position of the density critical surface α. Only a careful description of how the randomness is implemented in the model completely specifies the problem of damage spreading. In principle one could study the case of generic correlations among these random numbers. Here we consider only two cases: either all the r_j are independent [case (i), critical surface γ_i] or they are all identical [case (ii), critical surface γ_{ii}].

The evolution equation for the single site variable $\sigma = \sigma(i,t)$ is

$$\sigma' = [r_0 < p_0] \oplus ([r_1 < p_1] \oplus [r_0 < p_0])(\sigma_- \oplus \sigma_+) \oplus ([r_3 < p_3] \oplus [r_0 < p_0])\sigma_- \sigma_+ \tag{3}$$

In the case $p_0 = 0$, Eq. (3) assumes the simpler form

$$\sigma' = [r_1 < p_1](\sigma_- \oplus \sigma_+) \oplus [r_3 < p_3]\sigma_- \sigma_+ \tag{4}$$

and the correlations among the r_j do not affect the evolution of σ.

We can substitute the evolution equation for the replica $\eta = \sigma \oplus h$, with the evolution equation for the damage $h = \sigma \oplus \eta$, obtaining

$$\begin{aligned} h' = &([r_1 < p_1] \oplus [r_0 < p_0] \oplus ([r_3 < p_3] \oplus [r_0 < p_0])\,\sigma_+)\,h_- \\ &\oplus ([r_1 < p_1] \oplus [r_0 < p_0] \oplus ([r_3 < p_3] \oplus [r_0 < p_0])\,\sigma_-)\,h_+ \\ &\oplus ([r_3 < p_3] \oplus [r_0 < p_0])\,h_- h_+ \end{aligned} \tag{5}$$

This equation has the same structure as the evolution equation of the original model with $p_0 = 0$, Eq. (4). Remembering that only for this value of p_0 does the DK model exhibit a phase transition, we have a strong argument for the correspondence between directed percolation and damage-spreading transitions. However, also in the symmetric case $p_1 = p_2$ and $r_1 = r_2$, the evolution equation of h is symmetric only in average, and one has to take into consideration the correlations between σ_- and σ_+. As discussed before, these correlations can be included in the definition of the DK model, which specifies only the transition probabilities. It remains to be proved that all these versions do belong to the same universality class.

For the rest of this section we assume $p_0 = 0$. Previous numerical investigations showed that on this plane the two surfaces α and γ meet at the point $Q = (\sim 0.81, 0)$. Inserting the value $p_3 = p_0 = 0$ in Eq. (5), we see that the evolution law for h is the same of that for σ, and so both transitions coincide on this line. This corresponds also to the dilution of rule 90.

Since the rest of γ lies away from α, the correlations among sites decay rapidly in time and space. This allows us to use a mean-field approximation. In the simplest form, we replace $\sigma(i, t)$ with a random bit that assumes the value one with probability m. With this assumption Eq. (5) becomes

$$\begin{aligned} h' = &([r_1 < p_1] \oplus [r_3 < p_3][r_4 < m])\,h_- \\ &\oplus ([r_1 < p_1] \oplus [r_3 < p_3][r_5 < m])\,h_+ \oplus [r_3 < p_3]\,h_- h_+ \end{aligned} \tag{6}$$

where r_4 and r_5 are independent random numbers. This is a rather drastic approximation, both because of correlations and because the same $\sigma(i, t)$ is shared by $h(i-1, t+1)$ and $h(i+1, t+1)$. Nevertheless, we can assume this equation as a starting point in our derivation of the phase diagram.

We now want to remap this model onto the original DK model, assuming that the asymmetry ($r_4 \neq r_5$), which in average vanishes, does not strongly affect the transition.

The remapped transition probabilities $\tilde{p}$ are

$$\tilde{\tau}(0, 0 \to 1) = \tilde{p}_0 = 0$$

$$\tilde{\tau}(0, 1 \to 1) = \tilde{p}_1 = \pi([r_1 < p_1] \oplus [r_3 < p_3][r_5 < m])$$

$$\tilde{\tau}(1, 0 \to 1) = \tilde{p}_1 = \pi([r_1 < p_1] \oplus [r_3 < p_3][r_4 < m])$$

$$\tilde{\tau}(1, 1 \to 1) = \tilde{p}_3 = \pi([r_3 < p_3]([r_4 < m] \oplus [r_5 < m] \oplus 1))$$

where $\pi(f(r)) = \int_0^1 dr(r)$ is the probability that the Boolean function f of the random number r takes the value one.

For case (i) ($r_1 \neq r_3$), we have

$$\begin{aligned} \tilde{p}_1 &= p_1 + p_3 m - 2 p_1 p_3 m \\ \tilde{p}_3 &= p_3(1 - 2m(1-m)) \end{aligned} \tag{7}$$

while for case (ii) ($r_1 = r_3$)

$$\begin{aligned} \tilde{p}_1 &= m|p_1 - p_3| + (1-m)\, p_1 \\ \tilde{p}_3 &= p_3(1 - 2m(1-m)) \end{aligned} \tag{8}$$

Since γ lies in the $p_1 > p_3$ region, one has for case (ii)

$$\tilde{p}_1 = p_1 - m p_3$$

Notice that for $p_3 = 0$ or for $m = 0$, γ_i and γ_{ii} coincide, as already noticed numerically by Grassberger.[3]

Given a certain point (p_1, p_3), it belongs to γ [i.e., $H(p_1, p_3) = 0$] if the point $(\tilde{p}_1, \tilde{p}_3)$ belongs to α [i.e., $m(p_1, p_3) = 0$]. In order to draw the phase diagram for the Hamming distance, one has to know the value of the density m in all the parameter space, and in particular the position of α. Unfortunately, we do not have a simple expression for these quantities; in the next section we use an approximation in order to draw a rough phase diagram. However, we are able to demonstrate that α and γ are tangent at point Q.

The slope q of the normal to α at Q can be given as

$$q = \frac{\partial m}{\partial p_1} \Big/ \frac{\partial m}{\partial p_3}\bigg|_Q$$

Considering that $\gamma \equiv H(p_1, p_3) = 0 \equiv m(\tilde{p}_1(p_1, p_3), \tilde{p}_3(p_1, p_3)) = 0$, the partial derivatives of H are given by

$$\frac{\partial H}{\partial p_1} = \frac{\partial m}{\partial \tilde{p}_1}\frac{\partial \tilde{p}_1}{\partial p_1} + \frac{\partial m}{\partial \tilde{p}_3}\frac{\partial \tilde{p}_3}{\partial p_1}$$

$$\frac{\partial H}{\partial p_3} = \frac{\partial m}{\partial \tilde{p}_1}\frac{\partial \tilde{p}_1}{\partial p_3} + \frac{\partial m}{\partial \tilde{p}_3}\frac{\partial \tilde{p}_3}{\partial p_3}$$

One has to take into account that $\tilde{p}_j$ depends on p_i both directly and via m. Inserting the relations (7) or (8) and considering that at point Q, $m = p_3 = 0$, one obtains

$$q' = \frac{\partial H}{\partial p_1} \Big/ \left.\frac{\partial H}{\partial p_3}\right|_Q = q$$

Since we know from numerical experiments and from all the mean-field approximations except the very first one that the slope of α at Q is negative in the (p_1, p_3) plane, the tangency of γ to α implies a reentrant behavior for the damage transition curve, as observed in ref. 9.

3. THE PHASE DIAGRAM

The problem of sketching an approximate phase diagram for the damage in an analytical way has been dealed with by several authors.[7, 9, 10] Since any equation for the damage depends on the behavior of one replica, there are two sources of error to be controlled: the approximations for the evolution of one replica and that for the difference (or for the other replica). As a consequence, all approximation schemes proposed so far involve a large effort for a poor result. Our method is able to exploit the knowledge of the density phase to study the damage phase transition. There are several methods that rapidly converge to a good approximation of α; to our knowledge the best ones are the phenomenological renormalization group[5] and the cluster approximation (local structure)[18] improved by finite-size scaling. This latter method can also give a good approximation of the behavior of $m(p_0, p_1, p_3)$ at any point.

Since here we are not interested in numerical competitions, we use the high-quality data for the density transition line from ref. 3 combined with a first-order mean-field approximation for the density. The intersection of α with $p_0 = 0$ has been approximated by a fifth-order polynomial

$$p_3 = \sum_{i=0}^{5} a_i p_1^i \tag{9}$$

The simplest mean-field approximation for the asymptotic density m gives

$$m = \frac{1 - 2p_1}{p_3 - 2p_1} \tag{10}$$

By using these approximations one obtains from Eq. (7) or (8) the curves reported in Fig. 1, together with the presently best numerical results.[3] The main source of error is that in this mean-field approximation the surface α, eq. (9), does not corresponds to the zero of the density m, Eq. (10). This is particularly evident in the absence of reentrancy of γ_1 and γ_2. Nevertheless even this rough approximation is able to reproduce qualitatively the phase diagram and to exhibit the influence on the damage critical line of the different implementations of randomness. Notice that the damage curve from ref. 3 corresponds to the implementation of Eq. (7).

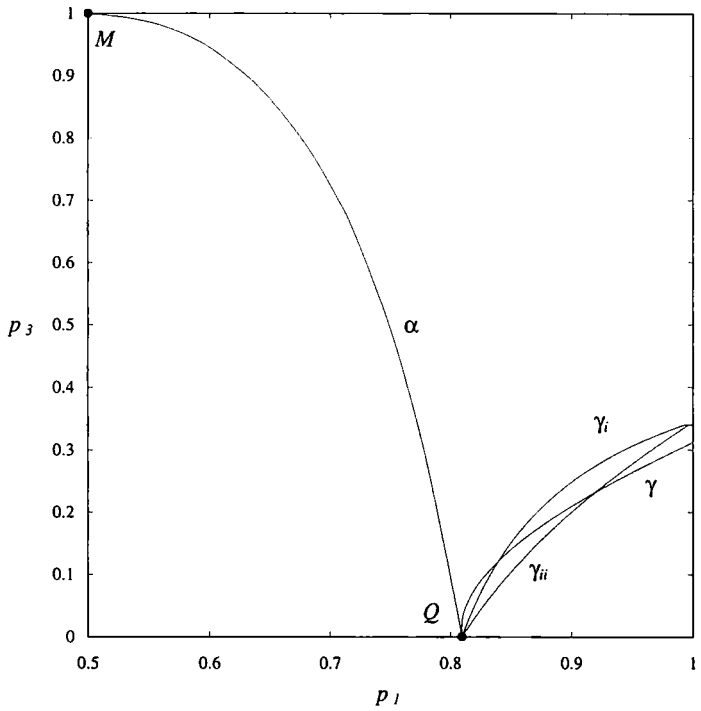

Fig. 1. Phase diagram for the density and the damage in the DK model for $p_0 = 0$. The curve labeled α is the density transition line and the one labeled γ is the damage transition line from ref. 3; the curves labeled γ_i and γ_{ii} correspond to mean-field approximations of Eqs. (7) and (8), respectively.

4. THE $p_0 > 0$ CASE

The DK model with arbitrary p_0 includes all one-dimensional symmetric cellular automaton models or spin systems with two inputs. We can represent each possible model as a point in the three-dimensional unit cube parametrized by p_0, p_1, p_3. The general form of the transition probabilities from Eq. (5) is

$$\begin{aligned} \tilde{p}_1 &= p_1 + (1-2p_1)(mp_3 + (1-m)\, p_0) \\ \tilde{p}_3 &= (p_0 + p_3 - 2p_0 p_3)(1 - 2m + 2m^2) \end{aligned} \tag{11}$$

There is a trivial transformation of the original DK model. One can invert ($0 \leftrightarrow 1$) all the spins before and after the application of the rule. The new transition probabilities p'_i are

$$\begin{aligned} p'_0 &= 1 - p_3 \\ p'_1 &= 1 - p_1 \\ p'_3 &= 1 - p_0 \end{aligned}$$

The critical plane $p_0 = 0$ maps to $p_3 = 1$, and the adsorbing state is now the configuration in which all spins are one. We indicate with the symbol α' the critical surface obtained by this transformation. The point Q is mapped to the point $Q' = (1, \sim 0.2, 1)$. The parameter cube and the critical curves are reported in Fig. 2. This mapping suggests the presence of a damaged zone near the corner (1, 0, 1).

In order to study the position of the critical surfaces for the damage, we numerically solved Eq. (11) combined with (9) in the very simple approximation $m = 0.5$. The results are reported in Fig. 2. Direct numerical simulations qualitatively agree with this picture.

The one-dimensional Ising model in zero field with heat bath dynamics can also be expressed with this formalism.

The local field $g = g_i$ for the one-dimensional Ising model is

$$g = K((2\sigma - 1) + (2\sigma_+ - 1))$$

where $K = \beta J = J/k_B T$ is the rescaled coupling constant and $\sigma = 0, 1$ the site variables (spin). The local field g can assume the values $-2K, 0, 2K$.

For the heat bath dynamics, σ' takes the value one with probability p given by

$$p = \frac{1}{1 + \exp(-2g)}$$

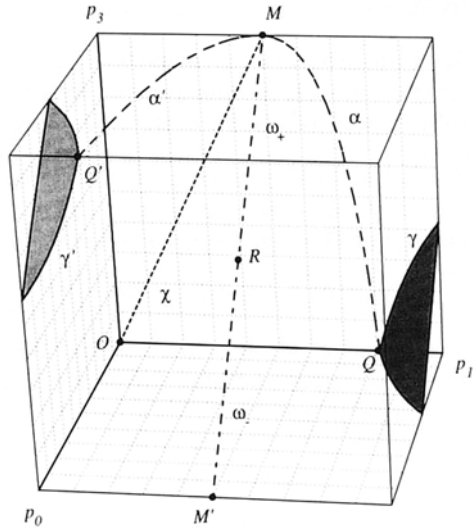

Fig. 2. The parameter cube for the general symmetric cellular automata. The dashed curves α and α' label the intersections of the corresponding surfaces with the planes $p_0 = 0$ and $p_3 = 1$, respectively, and correspond to the density phase transitions. The solid curves correspond to the intersection of the damage critical surfaces (shaded) γ and γ' with the boundaries of the cube. The dot-dashed lines labeled ω_+ and ω_- correspond to the existence line for the Ising model for positive and negative temperatures, respectively. The points M and M' label the critical points of the Ising model at zero temperature, and R corresponds to the limit of infinite temperature. The dotted line χ corresponds to the damage in the Ising model.

The transition probabilities are

$$p_0 = \frac{\xi}{1+\xi}$$

$$p_1 = \frac{1}{2}$$

$$p_3 = \frac{1}{1+\xi}$$

where $\xi = \exp(-4K)$. Notice that $p_3 = 1 - p_0$. For $T > 0$, we have $p_0 < 1/2$, while for negative temperatures, $p_0 > 1/2$. The point $p_0 = p_3 = 1/2$ corresponds to infinite T.

The evolution equation for the site variable is

$$\sigma' = [r < p_0] \oplus ([r < p_1] \oplus [r < p_0])(\sigma_- \oplus \sigma_+)$$
$$\oplus ([r < p_3] \oplus [r < p_0])\, \sigma_- \sigma_+$$

where usually all Taylor coefficients depend on the same random number $r = r(i, t)$. The existence line ω_+ for the Ising model with $T > 0$, $p_1 = 1/2$, $p_3 = 1 - p_0$, intersects α at $M = (0, 1/2, 1)$. The existence line ω_- for $T < 0$ ends at $M' = (1, 1/2, 0)$. The point $R = (1/2, 1/2, 1/2)$ corresponds to $T = \infty$ (see Fig. 2).

The evolution equation for the Hamming distance h is equivalent to Eq. (5) with all r_j equal to r. Taking into account the correlations induced by the random numbers and that the magnetization is 1/2 except at the critical point, one obtains

$$\tilde{p}_1 = \frac{1 - \xi}{2(1 + \xi)}$$

$$\tilde{p}_3 = \frac{1 - \xi}{1 + \xi}$$

i.e., the line $\chi \equiv (p_3 = 2p_2, p_0 = 0)$ for positive or negative temperatures. The line χ intersects α at point M for $T = 0^{\pm}$, confirming that the symmetry-breaking transition for the Ising model occurs at zero temperature.

5. CONCLUSIONS AND PERSPECTIVES

In this work we presented a formalism that allows the careful description of Boolean algorithms for stochastic cellular automata (including spin systems like the Ising model). Using this formalism, we were able to derive the exact equation for the evolution of a damage between two replicas that evolve under the same realization of the noise. Using a mean-field hypothesis, we gave strong indications that the critical line for the damage phase transition belongs to the same universality class as that for the density in the DK model, and thus as the directed percolation universality class. We mapped the density critical line to the damage critical line, obtaining the regions in the parameter space of a general symmetric cellular automaton where the replica symmetry breaking is to be expected. Our predictions are qualitatively confirmed by numerical simulations.

Several questions remain to be answered. Among others: Is it possible to obtain similar results starting from a field description? What does the phase diagram for more general (asymmetric, three-input, etc.) cellular automata look like?

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